

POLYNOMIAL SPLITTINGS OF OZSVÁTH AND SZABÓ'S d -INVARIANT

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ABSTRACT. For any rational homology 3-sphere and one of its spin^c -structures, Ozsváth and Szabó defined a topological invariant, called d -invariant. Given a knot in the 3-sphere, the d -invariants associated with the prime-power-fold branched covers of the knot, obstruct the smooth sliceness of the knot. These invariants bear some structural resemblances to Casson-Gordon invariants, which obstruct the topological sliceness of a knot. Se-Goo Kim found a polynomial splitting property for Casson-Gordon invariants. In this paper, we show a similar result for Ozsváth and Szabó's d -invariants. We give an application of the result.

1. INTRODUCTION

We work in smooth category, and all manifolds are supposed to be smooth unless stated otherwise. An oriented knot K in the 3-sphere S^3 is said to be *slice* if there is a smoothly embedded 2-disk Δ in the 4-ball B^4 satisfying $\partial(B^4, \Delta) = (S^3, K)$. Here Δ is called a *slice disk* of K . A pair of knots K_1 and K_2 are said to be *smoothly concordant* (which is denoted by $K_1 \sim K_2$) if $K_1 \# (-K_2)$ is slice where $-K_2$ is the mirror image of K_2 with reversed orientation. Smooth concordance is an equivalence relation among knots and the set of equivalence classes becomes an abelian group under the operation of connected sum. The group is called the knot concordance group and denoted by C . Slice knots represent the zero element in C .

A knot K is said to be *ribbon* if K bounds a smoothly immersed 2-disc in S^3 which has the property that the pre-image of each component of self-intersection consists of a properly embedded arc in the disc and an arc embedded in the interior of the disc. It is obvious that each ribbon knot is smoothly slice.

To study the group structure of C , there are two basic questions to consider. First, given a knot, we need to figure out what the order of the knot is in C . Second, given several knots K_1, K_2, \dots, K_n , we want to know if they are linearly independent or not in C . Finite order elements in C are called torsion elements. As for the first question, the only known torsion in C is 2-torsion, which comes from negative amphicheiral knots. Some invariants such as signature, Rasmussen invariant and Ozsváth and Szabó's τ -invariant, induce homomorphisms from C to the group of integers. If a knot is a torsion element, it has vanishing value on such invariants.

For the independence problem in C , there is a systematic way to study it by considering the relative primeness of the Alexander polynomials. Levine [9] first showed that if the connected sum of two knots with relatively prime Alexander polynomials has vanishing Levine obstructions, then so do both knots. Kim in [7] showed that the Casson-Gordon-Gilmer obstruction splits in the same manner. In [8], a similar splitting

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property was proved for von Neumann ρ invariants associated with certain metabelian representations.

In this paper, we study a similar splitting property for Ozsváth and Szabó's d -invariants, and apply this property to study the independence problem in C .

Given a 3-manifold Y and one of its torsion spin^c -structures s , the d -invariant $d(Y, s)$ is defined for (Y, s) by Ozsváth and Szabó in [11]. Let K be a knot in S^3 , and $\Sigma^n(K)$ be the n -fold cyclic branched cover of S^3 along K with $n = q^r$ for some prime number q . Then $\Sigma^n(K)$ is a rational homology 3-sphere. Therefore we can consider the d -invariant of $\Sigma^n(K)$ for any of its spin^c -structures.

If K is a slice knot, let $W^n(\Delta)$ be the n -fold branched cover of B^4 along a slice disk Δ of K . It is known that $W^n(\Delta)$ is a rational homology 4-ball whose boundary is $\Sigma^n(K)$. As studied in [11] and reformulated in many papers such as [3, 5], many of the d -invariants for $\Sigma^n(K)$ must vanish (see Theorem 2.2).

For any spin^c -structure s over $\Sigma^n(K)$, define

$$\bar{d}(\Sigma^n(K), s) = d(\Sigma^n(K), s) - d(\Sigma^n(K), s_0),$$

where s_0 is a special spin^c -structure being discussed in Section 2. The first homology group $H_1(\Sigma^n(K); \mathbb{Z})$ acts freely and transitively on the set of spin^c -structures $\text{Spin}^c(\Sigma^n(K))$. Given an element $s \in \text{Spin}^c(\Sigma^n(K))$ and an element $a \in H_1(\Sigma^n(K); \mathbb{Z})$, let $s + a$ denote the resulting element under the group action. We prove the following theorem.

Theorem 1.1. *Let K_1 and K_2 be two knots whose Alexander polynomials are relatively prime in $\mathbb{Q}[t, t^{-1}]$. Suppose further that at least one of K_1 and K_2 has non-singular Seifert form. Then the following hold.*

- (i) *If $K_1 \# K_2$ is slice, then for all but finitely many prime numbers q there exists a subgroup $M_i < H_1(\Sigma^n(K_i); \mathbb{Z})$ satisfying $|M_i|^2 = |H_1(\Sigma^n(K_i))|$ such that $\bar{d}(\Sigma^n(K_i), s_0 + m_i) = 0$ for any $m_i \in M_i$ and $i = 1, 2$, where n can be any power of q .*
- (ii) *If $K_1 \# K_2$ is ribbon, the conclusion holds for any prime number q .*

For the τ -invariant defined in [3], analogous properties can be proved by using the same argument. Furthermore, for invariants $\mathcal{T}_p^n(K)$ and $\mathcal{D}_p^n(K)$ which are defined similarly as $\mathcal{T}_p(K)$ and $\mathcal{D}_p(K)$ in [3], we have Theorem 2.7.

As an application of the results above, we show the following property, which has been known before [7].

Proposition 1.2. *Let T_k be the k -twist knot. Excluding the unknot, T_1 (which is the figure-8 knot) and T_2 (which is Stevedore's knot), no non-trivial linear combinations of twist knots are ribbon knots.*

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2. PROOF OF THEOREM 1.1

2.1. Alexander polynomial, Seifert form and d -invariant. In this subsection, we review some backgrounds needed in the proof of Theorem 1.1. Let K be a knot in S^3 with a Seifert surface F of genus g . Define the Seifert form $\theta : H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \rightarrow \mathbb{Z}$ as $\theta(x, y) := lk(i_+(x), y)$ for any simple closed curves x, y representing elements in $H_1(F; \mathbb{Z})$, where i_+ denotes the map that pushes a class off in the positive normal direction. Fix a basis $\{a_1, a_2, \dots, a_{2g}\}$ for $H_1(F; \mathbb{Z})$ and let A be the Seifert matrix associated with this basis.

A Seifert form θ on $H_1(F; \mathbb{Z})$ is said to be *null-concordant* if there exists a direct summand Z of $H_1(F; \mathbb{Z})$ so that $\text{rank}(Z) = \frac{1}{2}\text{rank}(H_1(F; \mathbb{Z}))$ and $\theta(Z, Z) = 0$. Such a direct summand Z is called a *metabolizer* of θ .

A knot which has a null-concordant Seifert form is called *algebraically slice*. Two knots K_1 and K_2 are said to be algebraically concordant if $K_1 \# (-K_2)$ is algebraically slice. The set of the equivalence classes of knots under this relation becomes a group as well, called *algebraic concordance group*, and we denote it C_{alg} .

The following lemma is Lemma 3.1 in [7], which was refined from [6, Proposition 3]. Note that the definition of null-concordance of a Seifert form in [7] is different from the one we are using, but the two definitions are equivalent.

Lemma 2.1 (Kervaire, Levine, Kim). *Given two knots K_1 and K_2 , let F_i be a Seifert surface of K_i , and θ_i be the Seifert form on $H_1(F_i; \mathbb{Z})$ for $i = 1, 2$. Suppose the Alexander polynomials of K_1 and K_2 are relatively prime in $\mathbb{Q}[t^{-1}, t]$, and either θ_1 or θ_2 is non-singular. Then if $\theta_1 \oplus \theta_2$ is null-concordant with a metabolizer Z , then θ_i is null-concordant with metabolizer $Z_i = Z \cap H_1(F_i; \mathbb{Z})$ for $i = 1, 2$.*

From this lemma we see that, with the assumption in this lemma, if $K_1 \# K_2$ is algebraically slice, so are both K_1 and K_2 .

Let $\Sigma^n(K)$ be defined as before for a given knot K and a prime power n . The homology group $H_1(\Sigma^n(K); \mathbb{Z})$ acts freely and transitively on the set $\text{Spin}^c(\Sigma^n(K))$. So a choice of one spin^c -structure gives a bijection between $\text{Spin}^c(\Sigma^n(K))$ and $H_1(\Sigma^n(K); \mathbb{Z})$. As discussed in Section 2 of [3], there exists a canonical spin^c -structure $s_0 \in \text{Spin}^c(\Sigma^n(K))$, which is uniquely characterised by K and n . For more details about the definition, please refer to [3] and [4]. If $H_1(\Sigma^n(K); \mathbb{Z})$ has no 2-torsion, then s_0 is the unique spin -structure over $\Sigma^n(K)$. Under this s_0 , we can identify $H_1(\Sigma^n(K))$ and $\text{spin}^c(\Sigma^n(K))$, by sending $m \in H_1(\Sigma^n(K); \mathbb{Z})$ to $s_0 + m \in \text{Spin}^c(\Sigma^n(K))$. One nice property of s_0 is that it is compatible with the connected sum of knots. Namely,

$$s_0(K_1 \# K_2) = s_0(K_1) \# s_0(K_2)$$

for two given knots K_1 and K_2 , where $s_0(K)$ denotes the spin^c -structure s_0 of $\Sigma^n(K)$.

If the knot K is slice, as before let $W^n(\Delta)$ be the n -fold branched cover of B^4 along a slice disk Δ of K . Consider the homomorphism $\zeta : H_1(\Sigma^n(K); \mathbb{Z}) \rightarrow H_1(W^n(\Delta); \mathbb{Z})$ induced by the inclusion map. Then it is known that a spin^c -structure over $\Sigma^n(K)$ can be extended to $W^n(\Delta)$ if and only if s has the form $s = s_0 + m$ for some $m \in \text{Ker}(\zeta)$. As an application of the results in [11], the following theorem is known (see also [3]).

Theorem 2.2 (Ozsváth and Szabó). *If K is slice, then $d(\Sigma^n(K), s_0 + m) = 0$ for any $m \in \text{Ker}(\zeta)$.*

2.2. Proof of Theorem 1.1. The order of $H_1(\Sigma^n(K); \mathbb{Z})$ is determined by the Alexander polynomial $\Delta_K(t)$ of K . Precisely we have Fox's formula

$$|H_1(\Sigma^n(K); \mathbb{Z})| = \left| \prod_{j=0}^{n-1} \Delta_K(\exp(2\pi j/n)) \right|.$$

We prove the following lemma.

Lemma 2.3. *Given a knot K and a prime number $p \geq 2$, let $S_{p,K}$ be the set of prime numbers with the property that if $q \in S_{p,K}$ then there is certain integer r such that p divides the order of $H_1(\Sigma^{q^r}(K); \mathbb{Z})$. Then $S_{p,K}$ is a finite set.*

Proof. Given a prime power n , consider the resultant of $\Delta_K(t)$ and $t^n - 1$ over the field of complex numbers \mathbb{C} . Then we have

$$\text{Res}(\Delta_K(t), t^n - 1) = a_\Delta^n \prod_{j=0}^{n-1} \Delta_K(\exp(2\pi j/n)),$$

where a_Δ is the leading coefficient of $\Delta_K(t)$. If p divides the order of $H_1(\Sigma^n(K); \mathbb{Z})$, then p divides $\text{Res}(\Delta_K(t), t^n - 1)$. Therefore $\text{Res}(\Delta_K(t), t^n - 1) = 0$ in the field \mathbb{F}_p , which means $\Delta_K(t)$ and $t^n - 1 = (t - 1)(t^{n-1} + t^{n-2} + \cdots + 1)$ having a common root in the algebraic closure of \mathbb{F}_p . Since 1 is never a root of $\Delta_K(t)$, we see that $\Delta_K(t)$ and $t^{n-1} + t^{n-2} + \cdots + 1$ have a common root in the algebraic closure of \mathbb{F}_p .

Since $p \geq 2$, we have $1 \notin S_{p,K}$. We assume that the set $S_{p,K}$ is an infinite set. Since $\Delta_K(t)$ has only finitely many roots in the algebraic closure of \mathbb{F}_p , there must be two elements q_1 and q_2 in $S_{p,K}$ such that $t^{q_1^{r_1}-1} + t^{q_1^{r_1}-2} + \cdots + 1$ and $t^{q_2^{r_2}-1} + t^{q_2^{r_2}-2} + \cdots + 1$ have a common root in the algebraic closure of \mathbb{F}_p for some positive integers r_1 and r_2 . This contradicts Lemma 2.4. Therefore opposed to our assumption, the set $S_{p,K}$ is a finite set. □

Lemma 2.4. *If m and n are relatively prime integers greater than or equal to two, $t^{m-1} + t^{m-2} + \cdots + 1$ and $t^{n-1} + t^{n-2} + \cdots + 1$ can never have a common root in the algebraic closure of \mathbb{F}_p .*

Proof. It is enough to check that $\text{Res}(t^{m-1} + t^{m-2} + \cdots + 1, t^{n-1} + t^{n-2} + \cdots + 1) = \pm 1$, which is non-zero in the algebraic closure of \mathbb{F}_p . In fact we have

$$\begin{aligned} (\dagger) \quad & \text{Res}(t^{m-1} + t^{m-2} + \cdots + 1, t^{n-1} + t^{n-2} + \cdots + 1) \\ & = \det(A(m, n)) = \pm 1, \end{aligned}$$

where

$$A(m, n) = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & \cdots & 0 & 0 & 0 & \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \\ \hline 1 & 1 & 1 & \cdots & 0 & 0 & 0 & \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \end{array} \right) \left. \begin{array}{l} n-1 \text{ rows,} \\ \text{with } m \text{ 1's on each row;} \\ \\ m-1 \text{ rows,} \\ \text{with } n \text{ 1's on each row.} \end{array} \right\}.$$

The first equation in (\dagger) is a basic property of the resultant of two polynomials. We prove the second equation by induction on $|m - n|$. If $|m - n| = 1$, we assume that $n = m + 1$ and subtract the j 's row from the $n + j - 1$'s row in $A(m, n)$, where $1 \leq j \leq m - 1$. Then we have

$$\det(A(m, n)) = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & \cdots & & & & \\ & 1 & 1 & \cdots & & & * & \\ & & \ddots & & & & & \\ 0 & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & 0 & & & & \ddots & \\ & & & & & & & 1 \end{array} \right) = 1.$$

If $|m - n| \geq 2$, assume $n > m$ and subtract the j 's row from the $n + j - 1$'s row, where $1 \leq j \leq m - 1$. The resulting matrix is

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & \cdots & & & & \\ & 1 & 1 & \cdots & & & * & \\ & & \ddots & & & & & \\ 0 & & & 1 & & & & \\ \hline & & & & 0 & & & \\ & & & & & A(m, n - m) & & \end{array} \right).$$

Therefore $\det(A(m, n)) = \det(A(m, n - m))$.

(1) If $|m - (n - m)| < n - m$, it follows from our induction that $\det(A(m, n)) = \det(A(m, n - m)) = \pm 1$.

(2) If $|m - (n - m)| \geq n - m$, then $2m - n \geq n - m \geq 2$, which implies $2n/3 \leq m < n$. Let k be the integer for which $kn/(k + 1) \leq m < (k + 1)n/(k + 2)$ for $k \geq 2$. Then we have

$$\begin{aligned} \det(A(m, n)) &= \det(A(m, n - m)) = \pm \det(A(n - m, 2m - n)) \\ &= \pm \det(A(n - m, 3m - 2n)) = \cdots \\ &= \pm \det(A(n - m, km - (k - 1)n)). \end{aligned}$$

The condition $kn/(k+1) \leq m < (k+1)n/(k+2)$ implies that $km - (k-1)n \geq n-m$ and $[km - (k-1)n - (n-m)] < (n-m)$. It follows from our induction that $\det(A(m, n)) = \pm \det(A(n-m, km - (k-1)n)) = \pm 1$. \square

Proof of Theorem 1.1. (i) Choose a Seifert surface F_i for the knot K_i for $i = 1, 2$. Let $K = K_1 \sharp K_2$ and $F = F_1 \sharp F_2$. Then F is a Seifert surface of K . Suppose K is a slice knot. Then $F \cup \Delta$ bounds a 3-manifold in B^4 , denoted by R , where Δ is a slice disk of K in the 4-ball B^4 . Consider the map $\iota : H_1(F; \mathbb{Z}) \rightarrow H_1(R; \mathbb{Z})/\text{Tor}$ induced by inclusion where Tor is the torsion part of $H_1(R; \mathbb{Z})$, and let $Z = \ker(\iota)$. Then Z is a metabolizer of the Seifert form θ associated with F . See Theorems 3.1.1 and 3.1.2 in [10] for detailed discussion.

Let Y denote S^3 sliced along F , and X denote D^4 sliced along R . Considering the construction of $\Sigma^n(K)$ and $W^n(\Delta)$, we have the following commutative diagram.

$$\begin{array}{ccccccc} \longrightarrow & \bigoplus_{1 \leq i \leq n} H_1(F; \mathbb{Z}) & \xrightarrow{j} & \bigoplus_{1 \leq i \leq n} H_1(Y; \mathbb{Z}) & \longrightarrow & H_1(\Sigma^n(K); \mathbb{Z}) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \bigoplus_{1 \leq i \leq n} H_1(R; \mathbb{Z}) & \longrightarrow & \bigoplus_{1 \leq i \leq n} H_1(X; \mathbb{Z}) & \longrightarrow & H_1(W^n(\Delta); \mathbb{Z}) & \rightarrow 0 \end{array}$$

The map j and the maps in the vertical direction are induced by the inclusion maps. We have the following isomorphisms.

$$H_1(Y; \mathbb{Z}) \cong H_1(S^3 - F; \mathbb{Z}) \cong H^1(F; \mathbb{Z}) \cong \text{Hom}(H_1(F; \mathbb{Z}), \mathbb{Z}) \cong H_1(F; \mathbb{Z}).$$

The second isomorphism follows from Alexander duality. In the same vein, we can establish the isomorphism $H_1(X; \mathbb{Z}) \cong H_1(R; \mathbb{Z})$. So we can replace the previous commutative diagram with the following diagram, which we denote by $(*)$.

$$\begin{array}{ccccccc} \longrightarrow & \bigoplus_{1 \leq i \leq n} H_1(F; \mathbb{Z}) & \xrightarrow{f} & \bigoplus_{1 \leq i \leq n} H_1(F; \mathbb{Z}) & \xrightarrow{g} & H_1(\Sigma^n(K); \mathbb{Z}) & \rightarrow 0 \\ & \downarrow & & \downarrow \bigoplus_{1 \leq i \leq n} \bar{\iota} & & \downarrow \zeta & \\ \longrightarrow & \bigoplus_{1 \leq i \leq n} H_1(R; \mathbb{Z}) & \longrightarrow & \bigoplus_{1 \leq i \leq n} H_1(R; \mathbb{Z}) & \xrightarrow{h} & H_1(W^n(\Delta); \mathbb{Z}) & \rightarrow 0 \end{array}$$

the maps $\bar{\iota}$ and ζ are induced by the inclusion maps.

We fix a basis $\{a_1, a_2, \dots, a_{2g}\}$ for $H_1(F; \mathbb{Z})$ and let A be the Seifert matrix of θ associated with this basis. Now we see there is a basis of $\bigoplus_{1 \leq i \leq n} H_1(F; \mathbb{Z})$ which is naturally induced by $\{a_1, a_2, \dots, a_{2g}\}$. Under this basis, the map f is represented by the matrix

$$f = \begin{pmatrix} G & I - G & 0 & 0 & \cdots & 0 \\ 0 & G & I - G & 0 & \cdots & 0 \\ 0 & 0 & G & I - G & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ I - G & 0 & 0 & 0 & \cdots & G \end{pmatrix},$$

where $G = (A^t - A)^{-1}A^t$ (See discussion before [2, Lemma 1] or [10, Theorem 6.2.2]). Here we abuse f to denote both the map and the matrix. It is known that f is a presentation matrix of $H_1(\Sigma^q(K); \mathbb{Z})$. Define f_1 and f_2 for K_1 and K_2 respectively. Then $f = f_1 \oplus f_2$.

Remember Z is the kernel of the map $\iota : H_1(F; \mathbb{Z}) \rightarrow H_1(R; \mathbb{Z})/\text{Tor}$. Let $M = g(\bigoplus_{1 \leq i \leq n} Z)$. By Lemma 2.5, the order of M is the square root of the order of $H_1(\Sigma^n(K); \mathbb{Z})$. By Lemma 2.1, the metabolizer Z has a splitting $Z = Z_1 \oplus Z_2$ where Z_i is a metabolizer of the Seifert form θ_i associated with F_i , for $i = 1, 2$. Then M has a splitting $M = M_1 \oplus M_2$ where $M_i = g(\bigoplus_{1 \leq i \leq n} Z_i)$ for $i = 1, 2$. By Lemma 2.5 the order of M_i is the square root of the order of $H_1(\Sigma^n(K_i); \mathbb{Z})$ for $i = 1, 2$.

Since $H_1(R; \mathbb{Z})$ is finitely generated, there are only finitely many prime numbers dividing the order of Tor , say they are elements in $\{p_1, p_2, \dots, p_s\}$. We let $S = \bigcup_{j=1}^s S_{p_j, K}$ where $S_{p_j, K}$ is the set described in Lemma 2.3, and then S is again a finite set.

Remember that $n = q^r$ for some prime number q and positive integer r . Suppose the prime number q is not in the set S . In this case, we claim that

$$h\left(\bigoplus_{1 \leq i \leq n} \bar{\iota}\left(\bigoplus_{1 \leq i \leq n} Z\right)\right) = h\left(\bigoplus_{1 \leq i \leq n} (\bar{\iota}(Z))\right) = 0.$$

Since Z is the kernel of the map $\iota : H_1(F; \mathbb{Z}) \rightarrow H_1(R; \mathbb{Z})/\text{Tor}$, so $\bar{\iota}(Z)$ belongs to the torsion part Tor of $H_1(R; \mathbb{Z})$. Given $x \in \bigoplus_{1 \leq i \leq n} Z$, let $y = \bigoplus_{1 \leq i \leq n} \bar{\iota}(x)$. Then the order of y divides the order of Tor . On the other hand, the order of $g(x)$ divides the order of $H_1(\Sigma^n(K); \mathbb{Z})$. By the commutativity of the diagram (*) above, $\zeta(g(x)) = h(y)$. If $\zeta(g(x)) = h(y) \neq 0$, the order of $h(y)$ divides both the orders of Tor and $H_1(\Sigma^n(K); \mathbb{Z})$. So there exists an element $p \in \{p_1, p_2, \dots, p_s\}$ dividing the order of $H_1(\Sigma^n(K); \mathbb{Z})$. This conflicts with the choice of q . Therefore $\zeta(g(x)) = h(y) = 0$, and our claim is proved.

By the commutativity of the diagram (*), we have $M \subset \text{Ker}(\zeta)$. Note that since K is slice, the order of $\text{Ker}(\zeta)$ is the square root of the order of $H_1(\Sigma^n(K); \mathbb{Z})$ by Lemma 3 in [1]. Therefore M and $\text{Ker}(\zeta)$ has the same order as finite groups, so $M = \text{Ker}(\zeta)$. The \mathfrak{d} -invariants defined on $M = \text{Ker}(\zeta)$ are zero by Theorem 2.2.

We now show that the $\bar{\mathfrak{d}}$ -invariants of $\Sigma^n(K_i)$ defined on M_i are zero for both $i = 1, 2$. For any element $m_1 \in M_1$, the element $(m_1, 0)$ is included in M , so by the additivity of \mathfrak{d} -invariant we have

$$d(\Sigma^n(K), s_0 + (m_1, 0)) = d(\Sigma^n(K_1), s_0 + m_1) + d(\Sigma^n(K_2), s_0) = 0.$$

Here we abuse s_0 to denote the unique spin^c -structures discussed in Section 2.1 over $\Sigma^n(K_1)$, $\Sigma^n(K_2)$ or $\Sigma^n(K)$. The value $d(\Sigma^n(K_1), s_0 + m_1) = -d(\Sigma^n(K_2), s_0)$ does not depend on the choice of m_1 , so we have

$$\bar{\mathfrak{d}}(\Sigma^n(K_1), s_0 + m_1) = d(\Sigma^n(K_1), s_0 + m_1) - d(\Sigma^n(K_1), s_0) = 0$$

for any $m_1 \in M_1$. The same fact can be proved for K_2 .

(ii) If K is a ribbon knot, we can choose R to be a handlebody, in which case $H_1(R; \mathbb{Z})$ is torsion free. Then the set S is an empty set. Therefore the conclusion in (i) holds for any prime power n .

□

In the rest of this section, we give a proof of the following lemma, which we cannot find a good reference for it.

Lemma 2.5. *Suppose Z is a metabolizer of the Seifert form θ for a Seifert surface F of the knot K . The order of $M = g(\bigoplus_{1 \leq i \leq n} Z)$ is the square root of the order of $H_1(\Sigma^n(K); \mathbb{Z})$ for any prime power n .*

Seifert [12] proved that $G^n - (G - I)^n$ is a presentation matrix for $H_1(\Sigma^n(K); \mathbb{Z})$ with the set of generators $\{a_1, a_2, \dots, a_{2g}\}$. Namely we have an exact sequence

$$0 \rightarrow H_1(F; \mathbb{Z}) \xrightarrow{\hat{f} := G^n - (G - I)^n} H_1(F; \mathbb{Z}) \xrightarrow{\hat{g}} H_1(\Sigma^n(K); \mathbb{Z}) \rightarrow 0.$$

The map \hat{g} induces an isomorphism $H_1(\Sigma^n(K); \mathbb{Z}) \cong H_1(F; \mathbb{Z})/\text{Im}(\hat{f})$ and $\hat{g}(Z)$ is isomorphic to $Z/(\text{Im}(\hat{f}) \cap Z)$.

We prove that the order of $Z/(\text{Im}(\hat{f}) \cap Z)$ is a square root of that of $H_1(F; \mathbb{Z})/\text{Im}(\hat{f})$. The proof is similar to that of [2, Lemma 2]. As stated there, we may extend a basis $\{x_1, x_2, \dots, x_g\}$ of Z to a basis $\{x_1, x_2, \dots, x_g, y_1, y_2, \dots, y_g\}$ of $H_1(F; \mathbb{Z})$. Under this basis, the matrices A , G and \hat{f} have the forms

$$A = \begin{pmatrix} 0 & C + I \\ C' & E \end{pmatrix}, G = \begin{pmatrix} C' + I & E' \\ 0 & -C \end{pmatrix} \text{ and } \hat{f} = \begin{pmatrix} C'_n & * \\ 0 & C_n \end{pmatrix},$$

where $C_n = C^n - (C - I)^n$. By the invertibility of C_n , we have $\text{Im}(\hat{f}) \cap Z \cong C'_n(Z)$. Therefore the order of $Z/(\text{Im}(\hat{f}) \cap Z)$ is $|\det(C'_n)|$, while the order of $H_1(F; \mathbb{Z})/\text{Im}(\hat{f})$ is $|\det(\hat{f})| = |\det(C'_n)|^2$.

Proof of Lemma 2.5. We show that $Z/(\text{Im}(\hat{f}) \cap Z)$ is isomorphic to $(\bigoplus_{1 \leq i \leq n} Z)/(\text{Im}(f) \cap \bigoplus_{1 \leq i \leq n} Z)$, which is isomorphic to $g(\bigoplus_{1 \leq i \leq n} Z)$. Following Lemma 1 in [2], there are integral determinant ± 1 $2gn \times 2gn$ matrices R and C which can be written as block matrices whose blocks are polynomials in G , such that $f^+ = RfC$ has the form

$$f^+ = \begin{pmatrix} I & * & \cdots & * & * \\ 0 & I & \cdots & * & * \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & * \\ 0 & 0 & \cdots & 0 & \hat{f} \end{pmatrix},$$

where the stars mean some unspecified polynomials in G . It is very easy to check that $(\bigoplus_{1 \leq i \leq n} Z)/(\text{Im}(f^+) \cap \bigoplus_{1 \leq i \leq n} Z) \cong Z/(\text{Im}(\hat{f}) \cap Z)$ by the forms of f^+ and \hat{f} .

Next we show that

$$(\bigoplus_{1 \leq i \leq n} Z)/(\text{Im}(f^+) \cap \bigoplus_{1 \leq i \leq n} Z) \cong (\bigoplus_{1 \leq i \leq n} Z)/(\text{Im}(f) \cap \bigoplus_{1 \leq i \leq n} Z)$$

by using the properties of R and C . Remember that R and C are automorphisms of $\bigoplus_{1 \leq i \leq n} H_1(F; \mathbb{Z})$, so $\text{Im}(f^+) = \text{Im}(RfC) = \text{Im}(Rf)$. Now we only need to show that R induces an isomorphism between $(\bigoplus_{1 \leq i \leq n} Z)/(\text{Im}(f) \cap \bigoplus_{1 \leq i \leq n} Z)$ and $(\bigoplus_{1 \leq i \leq n} Z)/(\text{Im}(Rf) \cap \bigoplus_{1 \leq i \leq n} Z)$. We show that

$$\begin{aligned} R(\bigoplus_{1 \leq i \leq n} Z) &= \bigoplus_{1 \leq i \leq n} Z, \text{ and} \\ (\dagger) \quad R(\text{Im}(f) \cap \bigoplus_{1 \leq i \leq n} Z) &\cong \text{Im}(Rf) \cap \bigoplus_{1 \leq i \leq n} Z. \end{aligned}$$

Choose an order for the elements in the basis of $\bigoplus_{1 \leq i \leq n} H_1(F; \mathbb{Z})$ so that the elements in $\bigoplus_{1 \leq i \leq n} Z$ take the first ng positions. Remember that the blocks of R are polynomials in G . The form of G tells us that under the reordered basis, the matrix R and its inverse are $2ng \times 2ng$ matrices with the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

where stars are $ng \times ng$ matrices. Since R and its inverse are automorphisms, it is now easy to check that relations (\ddagger) hold. □

2.3. τ -invariant, $\mathcal{T}_p^n(K)$ and $\mathcal{D}_p^n(K)$. Let \tilde{K} be the pre-image of K in $\Sigma^n(K)$. Considering \tilde{K} as a knot in $\Sigma^n(K)$, Grigsby, Ruberman and Strle in [3] defined the τ -invariant $\tau(\tilde{K}, s)$ for \tilde{K} and $s \in \text{Spin}^c(\Sigma^n(K))$. This invariant satisfies the following property.

Theorem 2.6 (Grigsby, Ruberman and Strle). *If K is slice, then $\tau(\tilde{K}, s_0 + m) = 0$ for any $m \in \text{Ker}(\zeta)$.*

Note that the proof of Theorem 1.1 only depends on the algebraic information carried on $\text{Ker}(\zeta)$ and Theorem 2.2, while does not depend on the definition of d -invariant. By replacing Theorem 2.2 with Theorem 2.6, we can prove exactly the same fact for τ -invariant as we did for d -invariant in Theorem 1.1.

Grigsby, Ruberman and Strle in [3] also defined invariants $\mathcal{D}_p(K)$ and $\mathcal{T}_p(K)$ associated with the double branched cyclic cover of the knot K . We can extend their definition naturally to the case of any n -branched cyclic cover with n a prime power.

Suppose $\phi : E \rightarrow \mathbb{Q}$ is a function on a finite abelian group E and $H < E$ is a subgroup. Following [3] we let $S_H(\phi) = \sum_{h \in H} (\phi(h))$. Given a prime number p , let \mathcal{G}_p be the set of all order p subgroups of $H_1(\Sigma^n(K); \mathbb{Z})$. Then we can define

$$\mathcal{T}_p^n(K) = \begin{cases} \min \left\{ \left| \sum_{H \in \mathcal{G}_p} n_H S_H(\tau(\tilde{K}, \cdot)) \right| \middle| \begin{array}{l} n_H \in \mathbb{Z}_{\geq 0} \text{ \& at least} \\ \text{one is non-zero} \end{array} \right\} & \begin{array}{l} \text{; if } p \text{ divides } |H_1(\Sigma^n(K); \mathbb{Z})| \\ \text{; otherwise} \end{array} \\ 0 & \end{cases}$$

and

$$\mathcal{D}_p^n(K) = \begin{cases} \min \left\{ \left| \sum_{H \in \mathcal{G}_p} n_H S_H(d(\Sigma^n(K), \cdot)) \right| \middle| \begin{array}{l} n_H \in \mathbb{Z}_{\geq 0} \text{ \& at least} \\ \text{one is non-zero} \end{array} \right\} & \begin{array}{l} \text{; if } p \text{ divides } |H_1(\Sigma^n(K); \mathbb{Z})| \\ \text{; otherwise} \end{array} \\ 0 & \end{cases}.$$

Here we regard $\tau(\tilde{K}, \cdot)$ and $d(\Sigma^n(K), \cdot)$ as functions from $H_1(\Sigma^n(K); \mathbb{Z})$ to \mathbb{Q} .

Given a function $\phi : E \rightarrow \mathbb{Q}$, we define $\bar{\phi} : E \rightarrow \mathbb{Q}$ by sending $e \in E$ to $\phi(e) - \phi(0)$. Let $\overline{\mathcal{T}}_p^n(K)$ and $\overline{\mathcal{D}}_p^n(K)$ be the invariants defined by taking $\bar{\tau}$ and \bar{d} . We prove the following theorem:

Theorem 2.7. *Let p be a positive prime number or 1. Suppose the Alexander polynomials of K_1 and K_2 are relatively prime in $\mathbb{Q}[t, t^{-1}]$. Suppose further that at least one of K_1 and K_2 has non-singular Seifert form.*

- (i) *If $n_1 K_1 \# n_2 K_2$ is a slice knot for some non-zero n_1 and n_2 , then for all but finitely many primes q , the following holds: $\overline{\mathcal{T}}_p^n(K_i) = \overline{\mathcal{D}}_p^n(K_i) = 0$ for $i = 1, 2$, where n is a power of q .*
- (ii) *If $n_1 K_1 \# n_2 K_2$ is a ribbon knot for some non-zero n_1 and n_2 , the conclusions above hold for any prime power n .*

Proof. The proof is a combination of the proof of Theorem 1.1, the proof of Theorem 1.2 in [3], Proposition 3.4 in [3], Theorem 2.2 and Theorem 2.6. \square

3. APPLICATION

It is known in [9, Corollary 23] that the twist knot T_k is

- of infinite order in the algebraic concordance group C_{alg} if $k < 0$;
- algebraically slice if $k \geq 0$ and $4k + 1$ is a square;
- of finite order in C_{alg} otherwise.

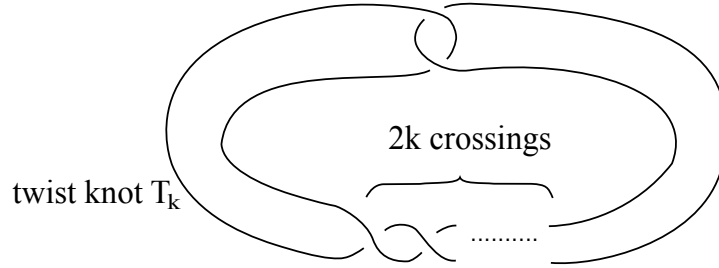


FIGURE 1.

Proof of Proposition 1.2. The Alexander polynomial of the T_k is

$$\Delta_{T_k}(t) = -kt^2 + (2k + 1)t - k.$$

It is easy to check that for any two non-trivial twist knots, their Alexander polynomials are relatively prime in $\mathbb{Q}[t^{-1}, t]$.

Note that each non-trivial twist knot has non-singular Seifert form. Excluding the unknot, 1-twist knot and 2-twist knot, suppose that there are two sets of positive integers $\{k_i\}_{i=1}^l$ and $\{n_i\}_{i=1}^l$ for which $K = \#_{i=1}^l (n_i T_{k_i})$ is a ribbon knot. Then

- by Lemma 2.1, each $n_i T_{k_i}$ is algebraically slice;
- by our Theorem 2.7, each T_{k_i} has vanishing $\overline{\mathcal{D}}_p^q$ and $\overline{\mathcal{T}}_p^q$ for any prime number p and prime number q .

We consider the case when $q = 2$, namely the double branched covers of the twist knots.

Recall that T_k has infinite order in the algebraic concordance group C_{alg} if $k < 0$. So each k_i for $1 \leq i \leq l$ is a positive integer.

Let L_k be the 3-manifold $\Sigma^2(T_k) = L(4k + 1, 2)$. Assume that $k \geq 0$ and let p be a prime number dividing $4k + 1$. Then

$$\overline{\mathcal{D}}_p^2(T_k) = \left| \sum_{j=0}^{p-1} \bar{d}(L_k, s_0 + j) \right| = \left| \sum_{j=0}^{p-1} (d(L_k, s_0 + j) - d(L_k, s_0)) \right|.$$

Ozsváth and Szabó [11] provided a formula of the d -invariants for lens spaces, by which we have

$$d(L_k, s_0 + j) = \frac{1}{4} - \frac{j^2}{8k + 2} + \begin{cases} \frac{1}{4} & \text{if } j \text{ is odd;} \\ -\frac{1}{4} & \text{if } j \text{ is even,} \end{cases}$$

for $0 \leq j \leq 2k$.

By calculation we have $d(L_k, s_0) = 0$. So

$$\overline{\mathcal{D}}_p^2(T_k) = \left| \sum_{j=0}^{p-1} d(L_k, s_0 + j) \right| = \mathcal{D}_p^2(T_k).$$

In [3, Proposition 5.1], the authors discussed $\mathcal{D}_p^2(T_k)$ for $k > 0$ and showed that

$$\mathcal{D}_p^2(T_k) > 0$$

except for the cases $k = 0, 1, 2$. Therefore those T_{k_i} which make $\overline{\mathcal{D}}_p^2$ vanishes are restricted to T_0, T_1 and T_2 . This completes the proof. \square

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